# Dynamical mean-field approximation for a pair contact process with a particle source

Attila Szolnoki*
*Research Institute for Technical Physics and Materials Science, P.O. Box 49, H-1525 Budapest, Hungary*
(Received 24 June 2002; published 11 November 2002)

The one-dimensional pair contact process with a particle source is studied by using dynamical cluster mean-field approximations with sites up to $n=12$. The results obtained for different levels of approximation become convergent especially for $n \geqslant 6$ and allow us to derive reliable extrapolations to the limit $n \to \infty$. At the zero source limit, the critical point exhibits a discontinuity whose magnitude vanishes with $1/n$. Coherent anomaly analysis of the data supports the conclusion that the vanishing of the order parameter and the density of isolated particles have the same critical behavior. In contrast to an earlier prediction, the present approximation does not support the existence of critical behavior in the inactive phase where the frozen density of isolated particles depends on the initial state.

DOI: 10.1103/PhysRevE.66.057102 PACS number(s): 05.70.Ln, 05.50.+q, 64.60.Cn

The study of phase transitions from active fluctuating phases into absorbing states has attracted considerable interest in the last decade [1,2]. The static exponents of these phase transitions belong frequently to the directed percolation (DP) universality class; however, there are also examples of other universality classes [3]. Open questions are related to the necessary conditions that can destroy this robust universal behavior. One of the simplest models showing static DP behavior is the pair contact process (PCP) which was proposed to realize a system involving infinitely many absorbing states [4]. Recently, Dickman *et al.* introduced a modified PCP model to explore the robustness of the DP transition [5]. In this model, each site of the one-dimensional lattice is either vacant or occupied by a single particle. A (randomly chosen) pair of nearest-neighbor particles is annihilated with a probability $p$ or an additional particle is created around the given pair with a probability $1-p$ if it is not forbidden by double occupancy. In the extended model, an external particle source is introduced that attempts to insert *isolated* particles with a rate $h$. This system exhibits an active phase when $p$ is smaller than a critical value $p_c$. For $p \geqslant p_c$ the system evolves into a frozen (absorbing) state where the nearest-neighbor pairs are absent. Further details of the model can be found in Ref. [5]. This extended model was studied by Monte Carlo (MC) simulations and the dynamical cluster mean-field approximation for quite large cluster sizes (ranged from $n=2$ to 6). Some disturbing behaviors, however, remained unsolved. For example, the analytical predictions do not tend monotonically toward the MC results for $h>0$ when the cluster size is increased. Furthermore, the analytical results shows a discontinuity in the variation of the critical point $p_c$ if $h \to 0$. This observation is surprising because the present approximation has proved to be satisfactory in many cases for $n \leqslant 6$ [6]. It is expected that further increase of the cluster size will resolve these discrepancies.

In this Brief Report, we discuss the results of dynamical mean-field approximations for cluster sizes as large as $n=12$. The present approach allows us to give more accurate extrapolations to the limit $n \to \infty$. Using this large number of data ($n=2-12$) we can also improve the predictions of the coherent anomaly method (CAM) [7] when considering the critical exponents around the transition point. Furthermore, we can study the frozen density of the isolated particle concentration when the system evolves toward the inactive phase [9].

To identify the transition point we have evaluated the $p$ dependence of the density of pairs of particles $\rho_2$ (henceforth considered as an order parameter) as well as the density of isolated particles $\rho_1$. The transition point is determined by the zero point of $\rho_2$ for $h=0$ and by the breaking point of the function $\rho_1(p)$ for $h>0$. This latter criterion was particularly useful because of the extremely low value of $\rho_2$ in a large interval of $p$.

We shall not describe the details of the mean-field technique because it has already been applied and demonstrated for the PCP model by several authors previously [5,9,10]. The derivation of the hierarchy of equations of motion for the configuration probabilities on the $n$-site clusters becomes complicated for large $n$. The technical difficulties increase drastically when enlarging the size of the clusters. Traditionally, the set of equations of motion is solved numerically in the stationary states. Despite the slow convergency toward the stationary solution for large $n$, the numerical integration of the master equations seems to be a more efficient method to find the stationary probability of configurations than the traditional Newton-Raphson method. Using the numerical integration method we determined all the configuration probabilities appearing on the 12-site cluster. (As an example, to get a data point in the last row of Table I requires four weeks running on a personal computer.) It is worth mentioning that very recently this method was used successfully for the consideration of a stochastic sandpile model [11].

Our results are plotted in Figs. 1–3. Table I summarizes the predictions at all levels of approximations for $p_c$ and for the density of isolated particles $\phi_{nat}$ at the critical point at different values of $h$.

Figure 1 demonstrates clearly that the predictions for $p_c$ become monotonically convergent for $n \geqslant 6$. The linear fit for $n>5$ data gives $p_c=0.075$ in close agreement with the MC data ($p_c^{MC}=0.077$). The same good convergence can be

*Electronic address: szolnoki@mfa.kfki.hu

 

TABLE I. Results of $n$-site approximations.

| $n$ | $p_c(h=0)$ | $p_c(h=0.1)$ | $p_c(h=10)$ | $\phi(h=0)$ | $\phi(h=0.1)$ | $\phi(h=10)$ |
|---|---|---|---|---|---|---|
| 2 | 0.2 | 0.66667 | 0.6667 | 0 | 0.5 | 0.5 |
| 3 | 0.12774 | 0.1820 | 0.182 | 0.23 | 0.4615 | 0.4615 |
| 4 | 0.11846 | 0.1502 | 0.1573 | 0.21665 | 0.5 | 0.5 |
| 5 | 0.11757 | 0.1440 | 0.1579 | 0.17989 | 0.4580 | 0.477 |
| 6 | 0.10272 | 0.1166 | 0.1353 | 0.22393 | 0.4383 | 0.4482 |
| 7 | 0.10069 | 0.1148 | 0.1323 | 0.21840 | 0.4450 | 0.4511 |
| 8 | 0.09692 | 0.1094 | 0.1265 | 0.22121 | 0.4390 | 0.4476 |
| 9 | 0.09387 | 0.1044 | 0.1224 | 0.22879 | 0.4356 | 0.4442 |
| 10 | 0.09218 | 0.1027 | 0.1198 | 0.22838 | 0.4351 | 0.4438 |
| 11 | 0.09048 | 0.1004 | 0.1175 | 0.23120 | 0.4325 | 0.4418 |
| 12 | 0.08925 | 0.0989 | 0.1156 | 0.23269 | 0.4316 | 0.4416 |
| $\infty$ | 0.075 | 0.080 | 0.095 | 0.242 | 0.422 | 0.433 |
| Simulation | 0.077091(5) | 0.086272(15) | 0.09785 | 0.241(1) | 0.421(1) | 0.433 |

obtained in the presence of external source. Comparisons of the linear fit with the MC data are also listed in Table I (MC data are taken from Ref. [5]).

Surprisingly, although the $n$-site approximations converge to the MC data at *arbitrary* values of $h$, a jump can be observed in $p_c$ when considering the limit $h\to 0$ for all $n$. The magnitude of this jump is defined as

$$\Delta p^n = \lim_{h\to 0} p_c^n(h) - p_c^n(h=0). \tag{1}$$

The inset of Fig. 1 shows $\Delta p^n$ for different levels of approximation. This log-log plot suggests that the jump decreases as a power law function, i.e., $\Delta p^n \propto n^{-\omega}$ with an exponent $\omega \approx 1.666$. The explanation of this value of $\omega$ remains to be clarified. This discontinuity of $\Delta p^n$ may also be observed for higher dimension versions of the model [12].

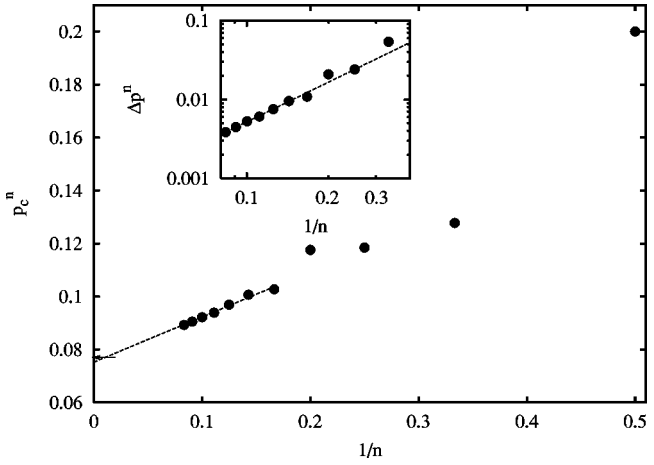


FIG. 1. Predictions for $p_c$ at different levels of approximation ($n=2,\ldots,12$) at $h=0$. The dashed line is the linear fit for $n=6,\ldots,12$ data and the arrow shows the MC data. Inset: Discontinuity in $p_c$ at $h\to 0$ as a function of the levels of approximation. $n=3,\ldots,12$–point levels are plotted. The dashed line corresponds to a power law with an exponent $\omega\approx 1.666$.

A distinct improvement of convergency can also be observed for higher values of $n$ if we consider the density of isolated particles $\phi_{nat}$ at $p_c$. The extrapolations, based on the predictions of $n>5$ data, are in excellent agreement with MC data.

Such a large number of approximations makes it possible to apply the CAM analysis introduced by Suzuki [7]. In the vicinity of the critical point the order parameter ($\rho_2$) and the density of isolated particles ($\rho_1$) are estimated by $\rho_2 \propto a_2^n(p_c^n - p)$ and $\rho_1 - \phi_{nat} \propto a_1^n(p_c^n - p)$, where $p_c^n$ denotes the prediction for the critical point at the $n$-point level. To estimate the $\beta$ critical exponents we have plotted the amplitudes of the mean-field results ($a_1^n$ and $a_2^n$) as a function of

$$\delta_n = (p_c^n/p_c)^{1/2} - (p_c/p_c^n)^{1/2}, \tag{2}$$

where $p_c$ denotes the result of the MC simulation. Figure 2 shows the mean-field amplitudes in the absence of an exter-

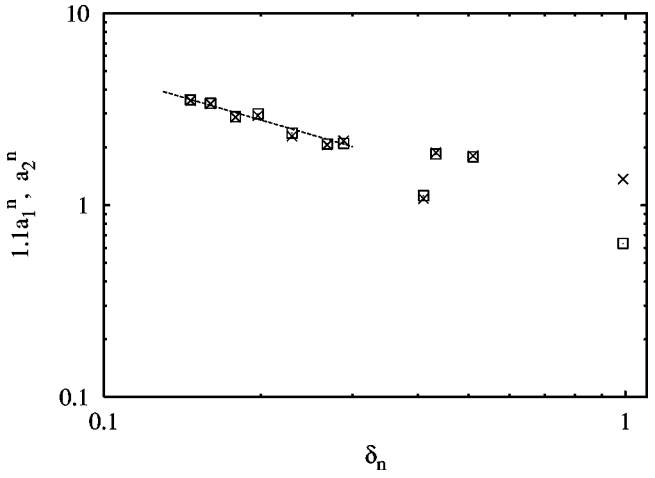


FIG. 2. CAM scaling of the critical mean-field coefficients for the order parameter ($a_2$, open squares) and the density of isolated particles ($a_1$, crosses). The dashed line corresponds to the $\beta = 0.276$ exponent. The values of $a_1^n$ are multiplied by 1.1.

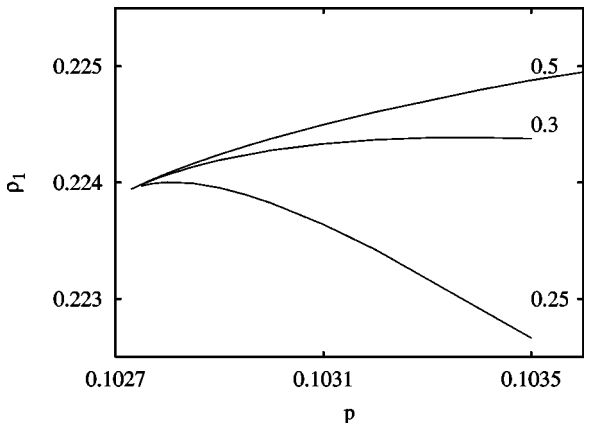


FIG. 3. The stationary value of particle concentration in the inactive phase at the six-point level at $h=0$. The initial concentrations for the different curves are indicated.

nal source ($h=0$) for different levels of approximation. Using different subsets of CAM-data the estimated exponents are 0.221, 0.246, 0.276, and 0.312. The dashed line corresponds to the $\beta=0.276$ DP exponent. Although the error bar is still large, the tendency to DP is straightforward. These results confirm earlier MC observations [8]; namely, the order parameter and the density of isolated particles can be described by the same power laws when the system approaches the critical point from the active phase. At the same time the CAM analysis of mean-field data helps to understand why the exponents agree. Figure 2 demonstrates that these two quantities are proportional to each other in the vicinity of the critical point. (In this plot $1.1a_1^n$ is compared with $a_2$.)

In the presence of a source ($h>0$) the earlier MC simulation suggested a slightly modified $\beta=0.287$ exponent [5]. Our mean-field data also become convergent for $n\geqslant 6$, suggesting an exponent close to that of the DP class ($\beta=0.274\pm 0.038$ at $h=10$). However, the region in $\delta_n$ is so small that the error bar of the estimated exponent is four times larger than the predicted change of the MC result. Therefore, this method is incapable of distinguishing such close exponents.

As noted above, the large-cluster approximation makes it possible to test the prediction of *static* critical behavior of particle concentration in the *inactive* phase [9]. Such a critical behavior of a static quantity in the inactive phase has already been observed in a different model [13]. In a previous MC study of the PCP model it was suggested that the density of isolated particles follows a power law $\rho_1^{nat}-\rho_1 \propto (p-p_c)^{\beta_1}$ for $p>p_c$ in the absence of an external source [9]. Following Marques *et al.* [9], we have calculated the absorbing state selected by the system's dynamics starting from the same homogeneous initial state. However, different initial particle concentrations were chosen to study the robustness of the final absorbing state in the inactive phase. The initial concentration ranges from $\phi_n$ to 0.99. The results of a six-point approximation are plotted in Fig. 3 for different values of initial concentration at $h=0$. It suggests that the absorbing state selected by the system's dynamics depends on the initial condition. Similar behavior may be observed for all levels of approximation. The observed deviation of curves increases further on increasing $p$ and the level of approximation. Obviously, the stationary value of the concentration becomes independent of the initial condition in the active phase.

We have tried to apply a CAM analysis to the data obtained from the *same* initial concentration for different levels of approximation. Even in this case, the irregularity of the mean-field coefficients at different levels of approximation does not allow one to extract the critical exponent. Briefly, the present approximation does not support the existence of static critical behavior in the inactive phase. In the light of this prediction further intensive MC simulations are suggested to clarify the possible existence of *natural absorbing states*.

In summary, we have demonstrated that the present dynamical mean-field approximations (for large $n$) yield adequate extrapolation to the limit $n\to\infty$ even in the presence of an external particle source that causes nonanalytical behavior in the limit $h\to 0$. According to this approximation the discontinuity of $p_c$ displays power law decay. The application of CAM analysis demonstrates that the densities of nearest-neighbor pairs and isolated particles are proportional to each other in the vicinity of the critical point and supports the same critical exponents of these quantities. This behavior is certainly related to the fact that the extinct pairs can leave extra solitary particles behind. Although our analysis has proved to be a useful tool for investigating stationary states in the active phase, it does not confirm the occurrence of an unambiguous critical behavior in the inactive phase where the composition of the frozen state depends on the initial state. We hope that the obscurity of the stationary inactive state will stimulate further MC simulations and theoretical investigations.

The author wishes to thank Ron Dickman, György Szabó, and M. C. Marques for stimulating discussions. This research was supported by the Hungarian National Research Fund (OTKA) under Grant No. F-30449 and Bolyai Grant No. (BO/0067/00).

[1] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University, Cambridge, England, 1999).

[2] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).

[3] H. K. Janssen, Z. Phys. B: Condens. Matter **42**, 151 (1981); P. Grassberger, *ibid.* **47**, 365 (1982).

[4] I. Jensen, Phys. Rev. Lett. **70**, 1465 (1993).

[5] R. Dickman, W. R. M. Rabêlo, and G. Ódor, Phys. Rev. E **65**, 016118 (2002).

[6] R. Dickman, Phys. Lett. A **122**, 463 (1987); P. L. Garrido, J. Marro, and R. Dickman, Ann. Phys. (N.Y.) **199**, 366 (1990); G. Szabó and I. Borsos, Phys. Rev. E **49**, 5900 (1994); J. J.

Alonso, P. L. Garrido, J. Marro, and A. Achahbar, J. Phys. A **28**, 4669 (1995); G. Ódor and A. Szolnoki, Phys. Rev. E **53**, 2231 (1996).
[7] M. Suzuki, J. Phys. Soc. Jpn. **55**, 4205 (1986); M. Kolesik and M. Suzuki, e-print cond-mat/9411109.
[8] G. Ódor, J. F. Mendes, M. A. Santos, and M. C. Marques, Phys. Rev. E **58**, 7020 (1998).
[9] M. C. Marques, M. A. Santos, and J. F. F. Mendes, Phys. Rev. E **65**, 016111 (2002); M. A. Muñoz, C. A. da Santos, and M. A. Santos, e-print cond-mat/0202244.
[10] E. Carlon, M. Henkel, and U. Schollwöck, Phys. Rev. E **63**, 036101 (2001).
[11] R. Dickman, e-print cond-mat/0204608.
[12] A. Szolnoki (unpublished).
[13] A. Lipowski and M. Droz, Phys. Rev. E **64**, 031107 (2001).